Spiral anchoring in anisotropic media with multiple inhomogeneities

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Abstract. Various PDE models have been suggested in order to explain and predict the dynamics of spiral waves in excitable media. In two landmark papers, Barkley noticed that some of the behaviour could be explained by the inherent Euclidean symmetry of these models. LeBlanc and Wulff then introduced forced Euclidean symmetry-breaking (FESB) to the analysis, in the form of individual translational symmetry-breaking (TSB) perturbations and rotational symmetry-breaking (RSB) perturbations; in either case, it is shown that spiral anchoring is a direct consequence of the FESB.

In this article, we provide a characterization of spiral anchoring when two perturbations, a TSB term and a RSB term, are combined, where the TSB term is centered at the origin and the RSB term preserves rotations by multiples of $\frac{2\pi}{\jmath^*}$, where $\jmath^* \geq 1$ is an integer. When $\jmath^* > 1$ (such as in a modified bidomain model), it is shown that spirals anchor at the origin, but when $\jmath^* = 1$ (such as in a planar reaction-diffusion-advection system), spirals generically anchor away from the origin.

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1. Introduction

Scientific literature has recorded the occurrences of spiral waves in excitable media, from the Belousov-Zhabotinsky chemical reaction to the electrical potential in cardiac tissue [1, 2, 4, 11, 16, 17, 22–24, 26, 32, 34, 36, 37]. This last instance provides an inspiration for the study of spiral wave, as they are believed to be a precursor to several fatal types of ventricular tachycardia and/or fibrillation. [18, 33, 34].

After Barkley realized that the resonant growth transition from rigid rotations to quasi-periodic meandering and drifting could be explained solely through the inherent Euclidean symmetries of excitable media (which were incorporated in the various models) [1–3], the theory of equivariant dynamical systems (together with the general center manifold reduction theorem of Sandstede, Scheel and Wulff [13, 27–30]) has been used on numerous occasions to explain and predict spiral wave dynamics [5–7, 10, 14, 15, 19, 20, 35].

One of the advantages of this approach is that it provides universal, model-independent explanations of observed dynamics and bifurcations of spiral waves that occur in *a priori* different situations; consider for instance the aforementioned Hopf bifurcation from rigid rotation to quasi-periodic meandering and drifting (observed in numerical simulations [4] and in actual chemical reactions [22]) or the anchoring/repelling of spiral waves on/from a site of inhomogeneity (seen in numerical integrations of an Oregonator system [24], in photo-sensitive chemical reactions [37] and in cardiac tissue [11]). Using a model-independent approach based on forced symmetry-breaking, it has been shown that anchoring/repelling of rotating waves generically takes place in systems in which the translation symmetry of $\mathbb{SE}(2)$ (the group of translation and rotation symmetries) is broken by a small perturbation [20], although the center of anchoring/repelling may not be an inhomogeneity when more than one such perturbations are present [7]. Similarly, some dynamics of spiral waves observed in

anisotropic media (*e.g.* phase-locking and/or linear drifting of meandering spiral waves) have been shown to be generic consequences of rotational symmetry-breaking [19,25,26].

Many of the phenomena in which spiral waves are observed experimentally are modeled by reaction-diffusion systems

$$\frac{\partial u}{\partial t} = D \cdot \nabla^2 u + f(u) \tag{1.1}$$

where u is a k-vector valued function of time and two-dimensional space, D is a matrix of diffusion coefficients and $f : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is a smooth reaction term. Their use as models is justified as Scheel has proved that systems of this form admit have time-periodic, rigidly rotating spiral wave solutions [31]. Such models are $\mathbb{SE}(2)-$equivariant: indeed, (1.1) is invariant under the transformations

$$u(t,x) \longmapsto u(t, x_1 \cos\,\theta - x_2 \sin\,\theta + p_1, x_1 \sin\,\theta + x_2 \cos\,\theta + p_2), \tag{1.2}$$

where $(\theta, p_1, p_2) \in \mathbb{S}^1 \times \mathbb{R}^2 \simeq \mathbb{SE}(2)$ and $x \in \mathbb{R}^2$ [12, 35]. But inhomogeneous media is not perfectly Euclidean: as a result, (1.1) does not provide an appropriate frame for the study of anisotropic media with inhomogeneities.

Consider rather the following generalization of the bidomain equations describing the electrical properties of anisotropic cardiac tissue

$$\begin{aligned} u_t &= \tfrac{1}{\varsigma}(u - \tfrac{u^3}{3} - v) + \nabla^2 u + \tfrac{\alpha\varepsilon}{1+\alpha(1-\varepsilon)}\Psi_{x_1x_1} \\ v_t &= \varsigma(u + \beta - \gamma v), \\ \nabla^2\Psi + \varepsilon g(\alpha,\varepsilon)\Psi_{x_2x_2} &= \varepsilon h(\alpha,\varepsilon)u_{x_2x_2}, \end{aligned} \tag{1.3}$$

where u is a transmembrane potential, v controls the recovery of the action potential, Ψ is an auxiliary potential (without obvious physical interpretation), x_1 is the preferred direction in physical space in which tissue fibers align, ε is a measure of that preference, g and h are appropriate model functions, and α, ς, β and γ are model parameters [5, 8, 21, 25].

If the tissue has equal anisotropy ratios (*i.e.* $\varepsilon = 0$), (1.3) decouples into the FitzHugh-Nagumo equations for u and v, and Poisson's equation for Ψ [25]. Ignoring the boundary, system (1.3) is $\mathbb{SE}(2)-$equivariant when $\varepsilon = 0$, while it is only $\mathbb{Z}_2\dot{+}\mathbb{R}^2-$equivariant otherwise (see [21, section 2.2] for details).

Cardiac tissue is littered with inhomogeneities. In order to make the analysis more tractable, let us make the modeling assumption that the inhomogeneities consist of a finite number of independent "sources" which are localized near distinct sites $\zeta_1, \ldots, \zeta_n$ in the plane (see [7] for a similar hypothesis). This situation could be modeled by the (slightly) perturbed bidomain equations

$$\begin{aligned}
u_t &= \frac{1}{\varsigma}(u - \frac{u^3}{3} - v) + \nabla^2 u + \frac{\alpha\varepsilon}{1+\alpha(1-\varepsilon)}\Psi_{x_1x_1} + \sum_{j=1}^{n} \mu_j g_j^u(\|x-\zeta_j\|^2, \mu) \\
v_t &= \varsigma(u + \beta - \gamma v) + \sum_{j=1}^{n} \mu_j g_j^v(\|x-\zeta_j\|^2, \mu), \\
\nabla^2\Psi &+ \varepsilon g(\alpha,\varepsilon)\Psi_{x_2x_2} = \varepsilon h(\alpha,\varepsilon) u_{x_2x_2},
\end{aligned} \tag{1.4}$$

where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ is a small parameter and $g_j^{u,v}$ are smooth functions, uniformly bounded in their variables.

The symmetry-breaking induced by anisotropy and the presence of inhomogeneities at the origin then yields an equivariant structure on the semi-flow $\Phi_{t,\varepsilon,\mu}$ of (1.4), the existence and uniqueness of which is shown in [9]. In particular, $\Phi_{t,\varepsilon,\mu}$

(E1) is $\mathbb{SE}(2)-$equivariant when $(\varepsilon, \mu) = 0$;

(E2) is $\mathbb{Z}_2\dot{+}\mathbb{R}^2-$equivariant when $\varepsilon \neq 0$ is small and $\mu = 0$;

(E3) preserves rotations around ζ_{i^*} (but generically not translations) when $\varepsilon = 0$ and all the μ_i are zero except μ_{i*}, and

(E4) is (generically) trivially equivariant when (ε, μ) is small and generic.

When $\varepsilon = 0$ and $n = 1$, LeBlanc and Wulff have shown that rotating waves generically anchor at (or are repelled by) the inhomogeneity [20]. When $\varepsilon = 0$ and

$n > 1$, Boily, LeBlanc and Matsui have shown that anchoring remains generic in parameter wedges, but that the anchoring/repelling is generically centered away from the inhomogeneities [7]. When $\mu = 0$, LeBlanc has shown that rotating waves have $\mathbb{Z}_2-$spatial symmetry [19].

There is no reason to believe that combining anisotropy with multiple inhomogeneities will change these results qualitatively: to wit, if a spiral anchors in non-anisotropic media, it should also do so in anisotropic media, but with $\mathbb{Z}_2-$spatial symmetry. However, the proofs of [7,19,20] cannot be pasted together to obtain this "obvious" result; some mathematical difficulties had to be overcome.

The goal of this paper, then, is to provide a detailed analysis of abstract dynamical systems which share properties (E1), (E3) and (E4), as well as some technical conditions which will be specified as we proceed, with the following slight modification:

(E2') the semi-flow is $\mathbb{Z}_{\jmath^*}\dot{+}\mathbb{R}^2-$equivariant when $\varepsilon \neq 0$ is small and $\mu = 0$, where $\jmath^*$ is some positive integer.

As noted, for the bidomain model we have $\jmath^* = 2$, and for a reaction-diffusion-advection system, we have $\jmath^* = 1$. The value of $\jmath^*$ will play an important role in the subsequent analysis. The paper is organized as follows. In the second section, we derive the center bundle equations of the semi-flow of an appropriate system near a hyperbolic rotating wave. We state and prove our main results for $n = 1$ in the third section: as long as $\jmath^* > 1$, spirals anchor qualitatively as they would in the absence of anisotropy. But in the case $\jmath^* = 1$, an unexpected result occurs: in a reaction-diffusion-advection system with one inhomogeneity, the tug of war between the advection current and the inhomogeneity ends in a tie, and spirals anchor at some point away from the inhomogeneity. To the best of our knowledge, this has yet to be observed either in simulations or in the lab. In the fourth section, we look at spiral anchoring in the presence of the most general form

of Euclidean symmetry-breaking. Finally, we perform a simple qualitative numerical experiment demonstrating the validity of our results.

2. Reduction to the Center Bundle Equations

Set $1 \leq \jmath^* \in \mathbb{N}$. Let X be a Banach space, $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ a neighborhood of the origin, and $\Phi_{t,\varepsilon,\mu}$ be a smoothly parameterized family (parameterized by $(\varepsilon, \mu) \in \mathcal{U}$) of smooth local semi-flows on X.

Let $\mathbb{SE}(2) = \mathbb{C} \dot{+} \mathbb{SO}(2)$ denote the group of all planar translations and rotations, and let

$$a : \mathbb{SE}(2) \longrightarrow \mathrm{GL}(X) \tag{2.1}$$

be a faithful and isometric representation of $\mathbb{SE}(2)$ in the space of bounded, invertible linear operators on X. For example, if X is a space of functions with planar domain, a typical $\mathbb{SE}(2)$ action (such as (1.2) in the preceding section) is given by

$$(a(\gamma)u)(x) = u(\gamma^{-1}(x)), \quad \gamma \in \mathbb{SE}(2).$$

We will parameterize $\mathbb{SE}(2)$ as follows: $\mathbb{SE}(2) \cong \mathbb{C} \times \mathbb{S}^1$, with multiplication given by $(p_1, \varphi_1) \cdot (p_2, \varphi_2) = (e^{i\varphi_1} p_2 + p_1, \varphi_1 + \varphi_2)$, $\forall\, (p_1, \varphi_1), (p_2, \varphi_2) \in \mathbb{C} \times \mathbb{S}^1$. For fixed $\xi \in \mathbb{C}$, define the following subgroups of $\mathbb{SE}(2)$:

$$\mathbb{C} \dot{+} \mathbb{Z}_{\jmath^*} = \left\{ \left(s, k\frac{2\pi}{\jmath^*} \right) \mid k \in \mathbb{Z},\ s \in \mathbb{C} \right\} \text{ and } \mathbb{SO}(2)_\xi = \{\, (\xi, 0) \cdot (0, \theta) \cdot (-\xi, 0) \mid \theta \in \mathbb{S}^1 \,\},$$

the latter of which is isomorphic to $\mathbb{SO}(2)$ and represents rotations about the point ξ. We will assume the following symmetry conditions on the family $\Phi_{t,\varepsilon,\mu}$ of semi-flows.

Hypothesis 1 *There exists n distinct points $\xi_1, \ldots, \xi_n$ in $\mathbb{C}$ such that if e_j denotes the*

j^{th} vector of the canonical basis in $\mathbb{R}^n$, then $\forall\, u \in X,\ \varepsilon \neq 0,\ \alpha \neq 0,\ t > 0,$

$$\begin{aligned}\Phi_{t,\varepsilon,0}(a(\gamma)u) = a(\gamma)\Phi_{t,\varepsilon,0}(u) &\iff \gamma \in \mathbb{C}\dot{+}\mathbb{Z}_{\jmath^*},\\ \Phi_{t,0,\alpha e_j}(a(\gamma)u) = a(\gamma)\Phi_{t,0,\alpha e_j}(u) &\iff \gamma \in \mathbb{SO}(2)_{\xi_j}, \quad \textit{and}\\ \Phi_{t,0,0}(a(\gamma)u) = a(\gamma)\Phi_{t,0,0}(u), &\quad \forall\,\gamma \in \mathbb{SE}(2).\end{aligned}$$

Hypothesis 1 basically states that $\Phi_{t,\varepsilon,\mu}$ satisfies properties (E1), (E2'), (E3), (E4). We are interested in the effects of the forced symmetry-breaking on normally hyperbolic rotating waves. Therefore, we will assume the following hypothesis.

Hypothesis 2 *There exists $u^* \in X$ and Ω^* in the Lie algebra of $\mathbb{SE}(2)$ such that $e^{\Omega^* t}$ is a rotation and $\Phi_{t,0}(u^*) = a(e^{\Omega^* t})u^*$ for all t. Moreover, the set $\{\,\lambda \in \mathbb{C} \mid |\lambda| \geq 1\,\}$ is a spectral set for the linearization $a(e^{-\Omega^*})D\Phi_{1,0}(u^*)$ with projection P_* such that the generalized eigenspace* $\mathrm{range}(P_*)$ *is three dimensional.*

In order to simplify the analysis, we only consider one-armed spiral waves, *i.e.* the isotropy subgroup of u^* in hypothesis 2 is trivial. It should be noted that hypotheses 1 and 2 hold for a large variety of spirals (such as decaying spirals), but not for all spirals (including Archimedean spirals) [31].‡

From now on, we assume $\Phi_{t,\varepsilon,\mu}$ is a semi-flow that satisfy both hypotheses, as well as all other hypotheses required in order for the center manifold theorems of [27–30] to hold. Then, for (ε, μ) near the origin in $\mathbb{R} \times \mathbb{R}^n$, the essential dynamics of the semi-flow $\Phi_{t,\varepsilon,\mu}$ near the rotating wave reduces to the following ordinary differential equations on the bundle $\mathbb{C} \times \mathbb{S}^1$ (see [5] for more details):

$$\begin{aligned}\dot{p} &= e^{i\varphi}\big[\nu + J^p(p,\overline{p},\varphi,\varepsilon,\mu)\big]\\ \dot{\varphi} &= \omega_{\mathrm{rot}} + J^{\varphi}(p,\overline{p},\varphi,\varepsilon,\mu),\end{aligned} \tag{2.2}$$

‡ Even when hypothesis 1 fails, finite-dimensional center-bundle equations which share the symmetries of the underlying abstract dynamical systems have a definite predictive value [1, 2, 19, 20, 22].

where ν is a complex constant, $0 \neq \omega_{\rm rot}$ is a real constant, $J^p(p, \overline{p}, \varphi, 0, 0) \equiv 0$ and $J^\varphi(p, \overline{p}, \varphi, 0, 0) \equiv 0$. Furthermore, the functions J^p and J^φ are smooth and uniformly bounded in p. If (ε, μ) is near the origin, we can re-scale time along orbits of (2.2), perform a simple computation and apply Taylor's theorem to get the following.

Proposition 2.1 *The symmetry conditions in hypothesis 1 imply that the equations* (*2.2*) *have the general form*

$$\dot{p} = e^{i\varphi(t)} \left[v + \varepsilon G(\varphi(t), \varepsilon) + \sum_{j=1}^{n} \mu_j H_j((p - \xi_j)e^{-i\varphi(t)}, \overline{(p - \xi_j)}e^{i\varphi(t)}, \mu_j) \right] \tag{2.3}$$

where, without loss of generality, $\varphi(t) = t$*,* $v \in \mathbb{C}$*,* $\mu = (\mu_1, \ldots, \mu_n)$*, and the functions* G, H_j *are smooth, periodic in* φ *and uniformly bounded in* p*, and* G *is* $2\pi/\jmath^*-$*periodic in* φ *for some positive integer* $\jmath^*$*.*§

As in [7], a $2\pi-$periodic solution $p_{\varepsilon,\mu}$ of (2.3) is called a *perturbed rotating wave* of (2.3). Define the average value

$$[p_{\varepsilon,\mu}]_{\rm A} = \frac{1}{2\pi} \int_0^{2\pi} p_{\varepsilon,\mu}(t)\, dt. \tag{2.5}$$

If the Floquet multipliers of $p_{\varepsilon,\mu}$ all lie within (resp. outside) the unit circle, we shall say that $[p_{\varepsilon,\mu}]_{\rm A}$ is the *anchoring* (resp. *repelling*, or *unstable anchoring*) *center* of $p_{\varepsilon,\mu}$.

§ Strictly speaking, the center bundle equations take the form

$$\dot{p} = \mathcal{F} + e^{it} \sum_{j \neq k} \lambda_j \lambda_k \mathcal{H}_{j,k}(p, \overline{p}, \xi_j, \xi_k, t, \lambda), \tag{2.4}$$

where $\mathcal{F}$ is the right hand side of (2.3), $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) = (\varepsilon, \mu_1, \ldots, \mu_n) \in \mathbb{R}^{n+1}$, and the functions $\mathcal{H}_{j,k}$ are suitably smooth and $2\pi-$periodic in t. However, the analysis in what follows is independent of the "mixed" perturbation terms $\mathcal{H}_{j,k}$, when $\|\lambda\|$ is small enough: in particular, it is then generically the case that if (2.4) has a $2\pi-$periodic solution, so does (2.3), and vice-versa, and these solutions share the same stability. The argument depends on the Taylor expansion of an the time-2π map (see [7] for an example of a similar analysis that includes the mixed perturbation terms).

In the following section, we perform an analysis of the anchoring of perturbed rotating waves of (2.3) for $n = 1$ and $\xi_1 = 0$ and parameter values (ε, μ) near the origin in $\mathbb{R}^2$. As noted, we will tackle the case $n > 1$ in section 4.

3. Analysis of the Center Bundle Equations $(n = 1, \xi_1 = 0)$

Equations (2.3) represent the dynamics near a normally hyperbolic rotating wave for a parameterized family $\Phi_{t,\varepsilon,\mu}$ of semi-flows satisfying the forced-symmetry breaking conditions in hypothesis 1. We start with a brief review of spiral anchoring in the two cases $\varepsilon = 0$ and $\mu_1 = 0$, which were studied in detail in [7, 19, 20], and then present the result of combining both types of perturbations.

3.1. The Case $\varepsilon = 0$

By writing $w = pe^{-it} + iv$, this system becomes

$$\dot{w} = -iw + \mu_1 \widetilde{H}(w, \overline{w}, \mu_1) \tag{3.1}$$

where $\widetilde{H}(w, \overline{w}, \mu_1) = H_1(w - iv, \overline{w} + i\overline{v}, 0, \mu_1)$. The following theorem is proved in [20].

Theorem 3.1 *Let $\alpha = \mathrm{Re}(D_1 \widetilde{H}(0, 0, 0))$, where $\tilde{H}$ is as in (3.1). If $\alpha \neq 0$, then for all $\mu_1 \neq 0$ small enough, (2.3) has a hyperbolic rotating wave*

$$p(t) = (-iv + O(\mu_1))\, e^{it}, \quad \varphi(t) = t. \tag{3.2}$$

The origin $[p]_{\mathrm{A}} = 0$ is an anchoring center if $a\mu_1 < 0$; it is a repelling center if $a\mu_1 > 0$.

In the case where the semi-flow $\Phi_{t,0,\mu_1}$ is generated by a system of planar reaction-diffusion partial differential equations, the solution (3.2) represents a wave which is rigidly and uniformly rotating around the origin in the plane. In the case where $\alpha\mu_1 < 0$, the rotating wave is locally asymptotically stable. When $\alpha\mu_1 > 0$, the rotating wave is unstable (see [24] for an experimental characterization of this phenomenon in an Oregonator model).

3.2. The Case $\mu_1 = 0$

Since G is $2\pi/\jmath^*$−periodic in φ (or in t, equivalently), it can be written as the uniformly convergent Fourier series

$$G(t,\varepsilon) = \sum_{m\in\mathbb{Z}} g_m(\varepsilon)e^{im\jmath^* t}. \tag{3.3}$$

The following theorem is proved in [19].

Theorem 3.2 *Let $\jmath^* > 1$, or $\jmath^* = 1$ and $g_{-1}(\varepsilon) \equiv 0$. Then, for all $\varepsilon \neq 0$ sufficiently small, the solutions of (2.3) are $2\pi/\jmath^*$−periodic in time with discrete $\mathbb{Z}_{\jmath^*}$−symmetry.*

In the case where the semi-flow $\Phi_{t,\varepsilon,0}$ is generated by a system of planar reaction-diffusion partial differential equations, the solutions of (2.3) represents discrete rotating waves in the physical space.

How do these two cases interact when they are combined? We shall see that the answer depends greatly on the nature of $\jmath^*$.

3.3. The Case $\jmath^* = 1$

Let $F_G : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ be defined by

$$F_G(t,\varepsilon) = e^{it}\Big[-iv + \varepsilon \sum_{m\neq -1} \frac{g_m(\varepsilon)e^{imt}}{i(m+1)}\Big], \tag{3.4}$$

where the g_m are the Fourier coefficients of G found in (3.3). Differentiating (3.4) yields

$$\dot{F}_G(t,\varepsilon) = e^{it}\big[v + \varepsilon G(t,\varepsilon) - \varepsilon g_{-1}(\varepsilon)\big]. \tag{3.5}$$

Generically, on the center bundle $\mathbb{C}\times\mathbb{S}^1$, the path $(F_G(t,\varepsilon), t)$ represents a discrete rotating wave around the origin with trivial spatio-temporal symmetry. Set $z = p - F_G$. Then, *via* (3.5), (2.3) is equivalent to

$$\dot{z} = \varepsilon g_{-1}(\varepsilon) + \mu_1 e^{it} H_1((z + F_G(t,\varepsilon))e^{-it}, \text{c.c.}, \mu_1). \tag{3.6}$$

Generically, $g_{-1}(\varepsilon) \neq 0$ in some neighbourhood of the origin. Thus, (3.6) does not generically have a periodic solution when $\mu_1 = 0$ and $\varepsilon \neq 0$. Furthermore, the analysis of (3.6) reduces to that of (3.1) when $\varepsilon = 0$ (see section 3.1 and [20]). Set $\varepsilon = \hat{\varepsilon}\mu_1$. Then (3.6) transforms to

$$\dot{z} = \mu_1 H_*(ze^{-it}, \overline{z}e^{it}, t, \hat{\varepsilon}, \mu_1), \tag{3.7}$$

where $H_*(w, \overline{w}, t, \hat{\varepsilon}, \mu_1) = \hat{\varepsilon} g_{-1}(\hat{\varepsilon}\mu_1) + e^{it} H_1(w + F_G(t, \hat{\varepsilon}\mu_1)e^{-it}, \mathrm{c.\,c.}, \mu_1)$ is smooth, $2\pi-$periodic in t and uniformly bounded in w.

Set $\alpha_1 = D_1 H(-iv, i\overline{v}, 0, 0)$. Near $(z, \hat{\varepsilon}, \mu_1) = (0, 0, 0)$, the time$-2\pi$ map P of (3.7) is given by

$$P(z, \hat{\varepsilon}, \mu_1) = z + 2\pi\mu_1 \Big[\hat{\varepsilon} g_{-1}(0) + \alpha_1 z + O\big(|z|^2\big) + O\big(\mu_1\big) + \text{higher order terms}\Big]. \tag{3.8}$$

The fixed points of (3.8) correspond to $2\pi-$periodic solutions of (3.7), and so to discrete rotating waves with trivial spatio-temporal symmetry in (2.3). Let

$$B_*(z, \overline{z}, \hat{\varepsilon}, \mu_1) = \hat{\varepsilon} g_{-1}(0) + \alpha_1 z + O\big(|z|^2\big) + O(\mu_1) + \text{higher order terms}$$

be the function inside the square brackets in (3.8). Note that $B_*(0, 0, 0, 0) = 0$ and that $D_1 B_*(0, 0, 0, 0) = \alpha_1$, which is generically not 0. By the implicit function theorem, there is a unique smooth function $z(\hat{\varepsilon}, \mu_1)$ defined near $(\hat{\varepsilon}, \mu_1) = (0, 0)$ with $z(0, 0) = 0$ and

$$B_*\big(z(\hat{\varepsilon}, \mu_1), \overline{z}(\hat{\varepsilon}, \mu_1), \hat{\varepsilon}, \mu_1\big) \equiv 0$$

near $z = 0$. However $z = 0$ is not in general a solution of $B_* = 0$ for small $(\hat{\varepsilon}, \mu_1)$, unless $g_{-1}(0) = 0$. Thus the generic behaviour of discrete rotating waves in (2.3) is to locally drift away from the origin. This leads to the following theorem.

Theorem 3.3 *Let α_1 and $g_{-1}(0) \neq 0$ be as in the preceding discussion, with* $\mathrm{Re}(\alpha_1) \neq 0$. *Then there exists a wedge-shaped region near $(\varepsilon, \mu_1) = (0, 0)$ of the form*

$$\mathcal{W} = \{(\varepsilon, \mu_1) \in \mathbb{R}^2 \,:\, |\varepsilon| < K|\mu_1|, \;\; K > 0, \;\; \mu_1 \textit{ near } 0\,\}$$

such that for all $(\varepsilon, \mu_1) \in \mathcal{W}$ *with* $\varepsilon \neq 0$, *(2.3) has a unique hyperbolic discrete rotating wave* $\mathcal{D}^1_{\varepsilon,\mu_1}$ *with trivial spatio-temporal symmetry and center* $[\mathcal{D}^1_{\varepsilon,\mu_1}]_{\mathrm{A}}$ *near, but generically not at, the origin. Furthermore,* $[\mathcal{D}^1_{\varepsilon,\mu_1}]_{\mathrm{A}}$ *is a center of anchoring when* $\mu_1 \operatorname{Re}(a_1) < 0$.

Proof: Let $z(\hat{\varepsilon}, \mu_1)$ be the unique continuous function solving the equation $B_* = 0$ for small parameter vectors $(\hat{\varepsilon}, \mu_1)$, as asserted above. When $\mu_1 = 0$, any z is a non-hyperbolic fixed point of P and so, from now on, we will assume that $\mu_1 \neq 0$. If that is the case, and if $\hat{\varepsilon}$ and μ_1 are small enough, the eigenvalues $\omega_{1,2}(\hat{\varepsilon}, \mu_1)$ of $DP(z(\hat{\varepsilon}, \mu_1), \hat{\varepsilon}, \mu_1)$ satisfy

$$|\omega_{1,2}(\hat{\varepsilon}, \mu_1)|^2 = 1 + 4\pi\mu_1 \operatorname{Re}(\alpha_1) + \mu_1 O(\hat{\varepsilon}, \mu_1) \neq 1,$$

since $\operatorname{Re}(\alpha_1) \neq 0$. When $\mu_1 \operatorname{Re}(\alpha_1) < 0$, the eigenvalues lie inside the unit circle and the fixed point is asymptotically stable; otherwise, it is unstable.

Let $K > 0$ be such that the preceding discussion holds whenever $\hat{\varepsilon} \in (-K, K)$ and let $\mathcal{W}$ be as stated in the hypothesis. If $(\hat{\varepsilon}, \mu_1)$ is such that the time-2π map (3.8) has a hyperbolic fixed point $z(\hat{\varepsilon}, \mu_1)$ near 0, then (2.3) has a hyperbolic $2\pi-$periodic orbit $\tilde{z}_{\hat{\varepsilon},\mu_1}(t)$ centered near the origin.

Let $0 \neq \hat{\varepsilon} \in (-K, K)$ and set $\varepsilon = \hat{\varepsilon}\mu_1$. Then $(\varepsilon, \mu_1) \in \mathcal{W}$, as $|\varepsilon| = |\hat{\varepsilon}||\mu_1| < K|\mu_1|$, and $\tilde{z}_{\hat{\varepsilon},\mu_1}(t)$ is a $2\pi-$periodic orbit for the pair (ε, μ_1), which we denote by $z_{\varepsilon,\mu_1}(t)$. Since $p = z - F_G$, (2.3) has a unique perturbed rotating wave $\mathcal{D}^1_{\varepsilon,\mu_1}$, with

$$[\mathcal{D}^1_{\varepsilon,\mu_1}]_{\mathrm{A}} = \frac{1}{2\pi}\int_0^{2\pi} (z_{\varepsilon,\mu_1}(t) - F_G(t, \varepsilon))\, dt = [z_{\varepsilon,\mu_1}]_{\mathrm{A}}.$$

But $[z_{\varepsilon,\mu_1}]_{\mathrm{A}} = O(\varepsilon, \mu_1)$ as $(\varepsilon, \mu_1) \to 0$ and so $\mathcal{D}^1_{\varepsilon,\mu_1}$ is a discrete rotating wave with trivial spatio-temporal symmetry and center near (but generically not at) the origin. The conclusion about the stability of $\mathcal{D}^1_{\varepsilon,\mu_1}$ follows directly from the hyperbolicity of the eigenvalues of DP. $\square$

When the parameter values stray outside of $\mathcal{W}$, all that can generically be said with certainty is that solutions of (2.3) locally drift away from the origin, and so anchoring cannot be centered there. After drifting, the spiral may very well get anchored at some point far from the origin, depending on the global nature of the function H_1 in (2.3).

The analysis of the situation near the $\varepsilon-$axis involves fixed points of (3.8) at ∞, which correspond to traveling waves in (2.3) [19, 20]. Such an analysis lies outside the scope of this paper; the situation will be investigated at a later date.

3.4. The Case $\jmath^* > 1$

Let $C^0_{\mathbb{R}}(\mathbb{C})$ and $C^1_{\mathbb{R}}(\mathbb{C})$ be the spaces of continuous and continuously differentiable functions from $\mathbb{R}$ to $\mathbb{C}$, respectively, and $\mathfrak{P}_t^{2\pi/\jmath^*}$ be the space of $2\pi/\jmath^*-$periodic functions of the variable t.

Then $C^0_{2\pi/\jmath^*} = \{f : f \in \mathfrak{P}_t^{2\pi/\jmath^*} \cap C^0_{\mathbb{R}}(\mathbb{C})\}$ and $C^1_{2\pi/\jmath^*} = \{f : f \in \mathfrak{P}_t^{2\pi/\jmath^*} \cap C^1_{\mathbb{R}}(\mathbb{C})\}$ are Banach spaces when endowed with the respective norms

$$||u||_0 = \sup\{|u(x)| : x \in [0, 2\pi/\jmath^*]\} \quad \text{and} \quad ||u||_1 = ||u||_0 + ||u'||_0,$$

and the linear operator $\mathcal{Y} : C^1_{2\pi/\jmath^*} \to C^0_{2\pi/\jmath^*}$ defined by $\mathcal{Y}(u) = iu + u'$ is bounded, invertible and has bounded inverse. The nonlinear operator $\mathcal{H}_G : C^1_{2\pi/\jmath^*} \times \mathbb{R}^2 \to C^0_{2\pi/\jmath^*}$ given by

$$\mathcal{H}_G(u, \varepsilon, \mu_1) = \mathcal{Y}(u) - \mu_1 H_1\left(u - iv + \varepsilon \sum_{m\in\mathbb{Z}} \frac{g_m(\varepsilon) e^{im\jmath^* t}}{i(m\jmath^* + 1)}, \text{c.c.}, \mu_1\right), \tag{3.9}$$

where the g_m are the Fourier coefficients of G found in (3.3), will play an important part in what follows. Note that $\mathcal{H}_G(0, 0, 0) = 0$ and $D_1\mathcal{H}_G(0, 0, 0) = i \neq 0$. Thus, by the implicit function theorem, there is a neighbourhood $\mathcal{N}$ of the origin in $\mathbb{R}^2$ and a unique smooth function $U : \mathbb{R}^2 \to C^1_{2\pi/\jmath^*}$ satisfying $U(0, 0) = 0$ and $\mathcal{H}_G(U(\varepsilon, \mu_1), \varepsilon, \mu_1) \equiv 0$ for all $(\varepsilon, \mu_1) \in \mathcal{N}$.

Let $F_G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ be defined by

$$F_G(t, \varepsilon, \mu_1) = e^{it}\Big[-iv + \varepsilon \sum_{m\in\mathbb{Z}} \frac{g_m(\varepsilon)e^{im\jmath^* t}}{i(m\jmath^*+1)} + U(\varepsilon, \mu_1)(t)\Big]. \tag{3.10}$$

Then

$$\mathcal{Y}(U(\varepsilon, \mu_1))(t) = \mu_1 H_1\left(F_G(t, \varepsilon, \mu_1)e^{-it}, \text{c.c.}, \mu_1\right) \tag{3.11}$$

and

$$\dot{F}_G(t, \varepsilon, \mu_1) = e^{it}\big[v + \varepsilon G(t, \varepsilon, \mu_1) + \mathcal{Y}\left(U(\varepsilon, \mu_1)\right)(t)\big]. \tag{3.12}$$

On the center bundle $\mathbb{C}\times GS^1$, the path $(F_G(t, \varepsilon, \mu_1), t)$ (generically) represents a discrete rotating wave around the origin with $\mathbb{Z}_{\jmath^*}-$spatio-temporal symmetry. Set $z = p - F_G$. Then, using (3.11) and (3.12), (2.3) rewrites as

$$\dot{z} = \mu_1 e^{it}\big[H_1((z + F_G(t, \varepsilon, \mu_1))e^{-it}, \text{c.c.}, \mu_1) - H_1(F_G(t, \varepsilon, \mu_1)e^{-it}, \text{c.c.}, \mu_1))\big], \tag{3.13}$$

which reduces to

$$\dot{z} = \mu_1 e^{it}\widehat{H}(ze^{-it}, \overline{z}e^{it}, t, \varepsilon, \mu_1), \tag{3.14}$$

where

$$\widehat{H}(w, \overline{w}, t, \varepsilon, \mu_1) = H_1(w + F_G(t, \varepsilon, \mu_1)e^{-it}, \text{c.c.}, \mu_1) - H_1(F_G(t, \varepsilon, \mu_1)e^{-it}, \text{c.c.}, \mu_1)$$

is $2\pi/\jmath^*-$periodic in t.

Near $(z, \varepsilon, \mu_1) = (0, 0, 0)$, the time$-2\pi$ map P of (3.14) is given by

$$P(z, \varepsilon, \mu_1) = z + 2\pi\mu_1\Big[(\alpha_1 + O(\varepsilon, \mu_1))z + O(\varepsilon, \mu_1)\overline{z} + O\big(|z|^2\big)\Big], \tag{3.15}$$

where α_1 is as defined in section 3.3. The fixed points of (3.15) correspond to $2\pi-$periodic solutions of (3.14), and so to discrete rotating waves with $\mathbb{Z}_{\jmath^*}-$spatio-temporal symmetry in (2.3). Let

$$B^*(z, \overline{z}, \varepsilon, \mu_1) = (\alpha_1 + O(\varepsilon, \mu_1))z + O(\varepsilon, \mu_1)\overline{z} + O\big(|z|^2\big)$$

be the function inside the square brackets in (3.15). Note that $B^*(0,0,0,0) = 0$ and that $D_1 B^*(0,0,0,0) = \alpha_1$, which is generically not 0. By the implicit function theorem, there is a unique smooth function $z(\varepsilon, \mu_1)$ defined near $(\varepsilon, \mu_1) = (0,0)$ with $z(0,0) = 0$ and

$$B^*\big(z(\varepsilon, \mu_1), \overline{z}(\varepsilon, \mu_1), \varepsilon, \mu_1\big) \equiv 0$$

near $z = 0$. But $z = 0$ is always a solution of $B^* = 0$; hence $z(\varepsilon, \mu_1) = 0$ for all small enough (ε, μ_1). This leads to the following theorem.

Theorem 3.4 *Let* α_1 *be as in the preceding discussion, with* $\mathrm{Re}(\alpha_1) \neq 0$. *If* (ε, μ_1) *is in a small deleted neighbourhood* $\mathcal{W}^{\jmath^*}$ *of the origin, (2.3) has a unique hyperbolic discrete rotating wave* $\mathcal{D}^{\jmath^*}_{\varepsilon,\mu_1}$ *with* $\mathbb{Z}_{\jmath^*}$*−spatio-temporal symmetry and center* $[\mathcal{D}^{\jmath^*}_{\varepsilon,\mu_1}]_A = 0$. *Furthermore,* $\mathcal{D}^{\jmath^*}_{\varepsilon,\mu_1}$ *is anchored at the origin if* $\mathrm{Re}(\alpha_1)\mu_1 < 0$.

4. Spiral Anchoring With General FESB

The case $n > 1$ combines the results from the preceding section and from section 3.2 in [7]; it provides a synopsis of spiral anchoring under the most general form of Euclidean symmetry-breaking. The results depend yet again on the nature of $\jmath^*$. The proofs follow the general lines of previous work and are thus omitted, in order to avoid unnecessary repetitions.

Theorem 4.1 *Let* $\jmath^* = 1$ *and* $k \in \{1, \ldots, n\}$. *Set* $\alpha_k = D_1 H_k(-iv, i\overline{v}, 0, 0)$. *Write* $\varepsilon = \mu_0$. *Generically, there is a wedge-shaped region of the form*

$$\mathcal{W}^1_k = \{(\mu_0, \mu_1, \ldots, \mu_n) \in \mathbb{R}^{n+1} \,:\, |\mu_j| < W_{j,k}|\mu_k|,\ \ W_{j,k} > 0,\ \text{for } j \neq k \text{ and } \mu_k \text{ near } 0\,\}$$

such that for all $(\varepsilon, \mu_1, \ldots, \mu_n) \in \mathcal{W}^1_k$ *with* $\varepsilon \neq 0$, *the center bundle equations (2.3) have a unique hyperbolic discrete rotating wave* $\mathfrak{D}^{1;k}_{\varepsilon,\mu}$, *with trivial spatio-temporal*

symmetry, centered away from ξ_k*. Furthermore,* $[\mathfrak{D}^{1;k}_{\varepsilon,\mu}]_{\mathrm{A}}$ *is a center of anchoring when* $\mu_k \operatorname{Re}(\alpha_k) < 0$*.*

Theorem 4.2 *Let* $\jmath^* > 1$ *and* $k \in \{1, \ldots, n\}$*. Set* $\alpha_k = D_1 H_k(-iv, i\overline{v}, 0, 0)$*. Generically, there is a cone-like region of the form*

$$\mathcal{W}^{\jmath^*}_k = \{(\varepsilon, \mu_1, \ldots, \mu_n) \in \mathbb{R}^{n+1} : |\varepsilon| < \varepsilon_0,\ |\mu_j| < W_{j,k}|\mu_k|,\ W_{j,k} > 0,\ \text{for } j \neq k \text{ and } \mu_k \text{ near } 0\}$$

such that for all $(\varepsilon, \mu) \in \mathcal{W}^{\jmath^*}_k$*, (2.3) has a unique hyperbolic discrete rotating wave* $\mathfrak{D}^{\jmath^*;k}_{\varepsilon,\mu}$ *with* $\mathbb{Z}_{\jmath^*}$*−spatio-temporal symmetry centered at* ξ_k*. Furthermore,* $[\mathfrak{D}^{\jmath^*;k}_{\varepsilon,\mu}]_{\mathrm{A}}$ *is a center of anchoring when* $\mu_k \operatorname{Re}(\alpha_k) < 0$*.*

5. Numerical Simulations

In this section, we examine a system that models excitable media with anisotropy and a single inhomogeneity, which give rise to a semi-flow satisfying the FESB equivariance conditions described by (E1), (E2'), (E3) and (E4) for $\jmath^* = 1$.

The computations are carried out on a two-dimensional square domain‖ with Neumann boundary condition, using a 5-point Laplacian and the Runge-Kutta forward integrator of order 2. This naive numerical approach is used instead of a more robust algorithm mostly because the aim of this section is to illustrate the qualitative (rather than quantitative) power of theorems 4.1 and 4.2.

Given appropriate initial conditions, this system sustains spiral waves. A planar reaction-diffusion-advection system (RDAS) is a reaction-diffusion system to which an advection term has been added:

$$u_t = \tilde{D}\Delta u + \tilde{A}_1 u_{x_1} + \tilde{A}_2 u_{x_2} + f(u), \tag{5.1}$$

where $\tilde{A}_i \in \mathbb{M}_2(\mathbb{R})$ for $i = 1, 2$, and the other terms are as in (1.1).

‖ More precisely, on the domain $[-30, 30]^2$ with 200 grid points to a side and time-step $\Delta t = 0.005$.

Advection terms are used to model a directed flow or current through the excitable medium under consideration; for instance, the modified RDAS

$$u_t = \tilde{D}\Delta u + f(u) + \varepsilon\left[\tilde{A}_1(\varepsilon,\mu)u_{x_1} + \tilde{A}_2(\varepsilon,\mu)u_{x_2}\right] + \mu g(u, \|x-\xi\|^2, \varepsilon, \mu), \qquad (5.2)$$

where $(\varepsilon,\mu) \in \mathbb{R}^2$ is small, $\tilde{A}_1, \tilde{A}_2 : \mathbb{R}^2 \to \mathbb{M}_2(\mathbb{R})$ are smooth functions and g is a smooth bounded function, could model membrane potentials in a piece of cardiac tissue subjected to a directed current, with an inhomogeneity located at ξ.

Let $\Phi_{t,\varepsilon,\mu}$ denote the semi-flow generated by (5.2): it is $\mathbb{SE}(2)-$equivariant when $(\varepsilon,\mu) = 0$; $\mathbb{C}\dot{+}\{1\}-$equivariant when $\varepsilon \neq 0$ is small and $\mu = 0$; $\mathbb{SO}(2)_\xi-$equivariant when $\mu \neq 0$ is small and $\varepsilon = 0$, and (generically) trivially equivariant when (ε,μ) is small and generic.¶

However, it is not clear if theorem 3.3 can be applied to (5.2) since it is not yet known if the center manifold reduction theorems of [27–30] hold in general for RDAS; as such, the center bundle equations need not be of the form (2.2) in the presence of advection.

Yet, the following example shows that the conclusion of theorem 3.3 holds for the following RDAS, and so that, in some sense, (2.2) may capture the essential dynamics near a rotating wave when the semi-flow of the PDE has the general symmetry-breaking properties outlined above.

Consider the RDAS

$$\begin{aligned} u_t &= \tfrac{1}{\varsigma}\left(u - \tfrac{1}{3}u^3 - v\right) - 3\sqrt{2}\sin\left(\tfrac{0.03\pi}{2}\right)\phi_2 + \Delta u + 0.002u_{x_1}, \\ v_t &= \varsigma(u + \beta - \gamma v + \phi_2), \end{aligned} \qquad (5.3)$$

where $\varsigma = 0.3$, $\beta = 0.6$, $\gamma = 0.5$ and ϕ is defined *via*

$$\phi(x) = 0.12 f(x_1 + 10, x_2 - 5\sqrt{3}),$$

¶ This can be seen by slightly modifying the proof of theorem 2.1.3 in [5].

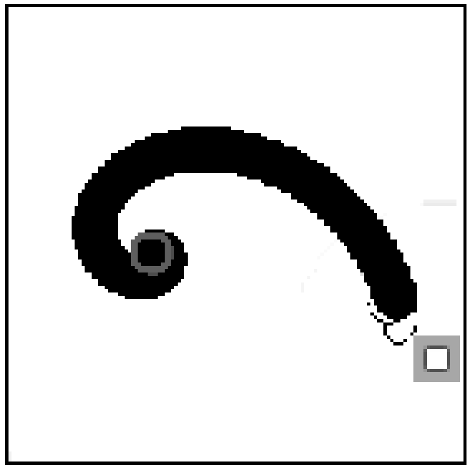

Figure 1. Anchoring in the RDAS equations (5.3). The spiral tip path is plotted in black. The anchored perturbed rotating wave is shown as a gray closed curve; the gray square indicates the location of the perturbation center.

with $f(x) = \exp\left(-0.00086\left(x_1^2 + x_2^2\right)\right)$. A single integration (see figure 1) clearly shows the spiral anchoring away from the perturbation center.

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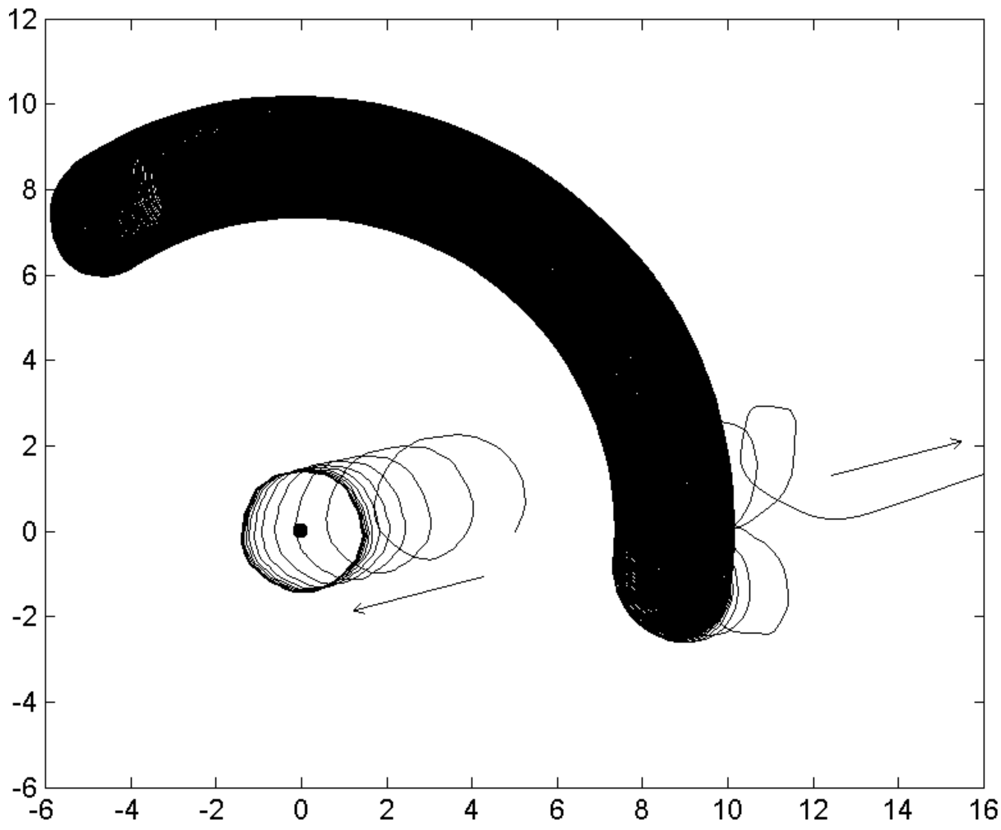

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